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# GLUEBALL MASSES IN $(2+1)$ -DIMENSIONAL ANISOTROPIC WEAKLY-COUPLED YANG-MILLS THEORY

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## Abstract

The confinement problem has been solved in the anisotropic  $(2+1)$ -dimensional  $SU(N)$  Yang-Mills theory at weak coupling. In this paper, we find the low-lying spectrum for  $N = 2$ . The lightest excitations are pairs of fundamental particles of the  $(1+1)$ -dimensional  $SU(2) \times SU(2)$  principal chiral nonlinear sigma model bound in a linear potential, with a specified matching condition where the particles overlap. This matching condition can be determined from the exactly-known S-matrix for the sigma model.

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# 1 Introduction

In recent papers, some new techniques have been developed for calculating quantities in a  $(2+1)$ -dimensional  $SU(N)$  gauge theories [1], [2], [3]. These techniques exploit the fact that in an anisotropic limit of small coupling, the gauge theory becomes a collection of completely-integrable quantum field theories, namely  $SU(N) \times SU(N)$  principal chiral nonlinear sigma models. These integrable systems are decoupled, save for a constraint which is necessary for complete gauge invariance. In the case of  $N = 2$ , it is possible to perturb away from integrability, using exactly-known off-shell matrix elements of the integrable theory.

Though the gauge theory we consider is not spatially-rotation invariant, it has features one expects of real  $(3+1)$ -dimensional QCD; it is asymptotically free and confines quarks at weak coupling. Thus the limit of no regularization is accessible.

One can formally remove the regulator in strong-coupling expansions of  $(2 + 1)$ -dimensional gauge theories; the vacuum state in this expansion yields a string tension and a mass gap which have formal continuum limits. This can be done in a Hamiltonian lattice formalism [4], or with an ingenious choice of degrees of freedom and point-splitting regularization [5]. This leaves open the question of whether these expressions can be trusted at weak coupling (more discussion of this issue can be found in the introduction of reference [2]). In particular, one would like to rule out a deconfinement transition, or different dependence of physical quantities on the coupling (as in compact QED [6]). There is a proposal for the vacuum state [7], in the formulation of reference [5] which seems to give correct values for some glueball masses [8], but this proposal evidently requires more mathematical justification.

In this paper, we will work out the masses of the lightest glueballs for the case of gauge group  $SU(2)$ . Our method would also work in principle for  $SU(N)$  gauge theories, and our reason for choosing  $N = 2$  is that the analysis is simplest for that case.

The basic connection between the gauge theory and integrable systems is most easily seen in axial gauge [1]. The string tensions in the  $x^1$ -direction and  $x^2$ -direction (which we called the horizontal and vertical string tensions, respectively) for very small  $g'_0$ , were found by simple physical arguments. The result for the horizontal string tension was confirmed for gauge group  $SU(2)$ , and additional corrections in  $g'_0$  were found [2], through the use of exact form factors for the currents of the sigma model. String tensions for higher representations can also be worked out, and adjoint sources are not confined [3].

Careful derivations of the connection between the gauge theory and integrable systems use the Kogut-Susskind lattice formalism [1], [2]. A shorter derivation was given in reference [9], which we summarize again here. The formalism is essentially that of “deconstruction” [10].

The Yang-Mills action is  $\int d^3\mathcal{L}$ , where the Lagrangian is  $\mathcal{L} = \frac{1}{2e'^2} \text{Tr} F_{01}^2 + \frac{1}{2e^2} \text{Tr} F_{02}^2 - \frac{1}{2e^2} \text{Tr} F_{12}^2$ , and where  $A_0, A_1$  and  $A_1$  are  $SU(N)$ -Lie-algebra-valued components of the

gauge field, and the field strength is  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ . This action is invariant under the gauge transformation  $A_\mu(x) \rightarrow ig(x)^{-1}[\partial_\mu - iA_\mu(x)]g(x)$ , where  $g(x)$  is an  $SU(N)$ -valued scalar field. We take  $e' \neq e$ , thereby losing rotation invariance.

We discretize the 2-direction, so that the  $x^2$  takes on the values  $x^2 = a, 2a, 3a \dots$ , where  $a$  is a lattice spacing. All fields are considered functions of  $x = (x^0, x^1, x^2)$ . We define the unit vector  $\hat{2} = (0, 0, 1)$ . We replace  $A_2(x)$  by a field  $U(x)$  lying in  $SU(N)$ , via  $U(x) \approx \exp -iaA_2(x)$ . There is a natural discrete covariant-derivative operator:  $\mathcal{D}_\mu \mathcal{U}(x) = \partial_\mu \mathcal{U}(x) - iA_\mu(x)\mathcal{U}(x) + i\mathcal{U}(x)A_\mu(x + \hat{2}a)$ ,  $\mu = 0, 1$ , for any  $N \times N$  complex matrix field  $\mathcal{U}(x)$ . The action is  $S = \int dx^0 \int dx^1 \sum_{x^2} a \mathcal{L}$  where

$$\mathcal{L} = \frac{1}{2(g'_0)^2 a} \text{Tr} F_{01}^2 + \frac{1}{2g_0^2} \text{Tr} [\mathcal{D}_0 U(x)]^\dagger \mathcal{D}_0 U(x) - \frac{1}{2g_0^2} \text{Tr} [\mathcal{D}_1 U(x)]^\dagger \mathcal{D}_1 U(x) , \quad (1.1)$$

and where  $g_0^2 = e_0^2 a$  and  $(g'_0)^2 = e'^2 a$ . The Lagrangian (1.1) is invariant under the gauge transformation:  $A_\mu(x) \rightarrow ig(x)^{-1}[\partial_\mu - iA_\mu(x)]g(x)$  and  $U(x) \rightarrow g(x)^{-1}U(x)g(x + \hat{2}a)$  where again,  $g(x) \in SU(N)$  and  $\mu$  is restricted to 0 or 1. The bare coupling constants  $g_0$  and  $g'_0$  are dimensionless. We recover from (1.1) the anisotropic continuum action in the limit  $a \rightarrow 0$ . The sigma model field is  $U(x^0, x^1, x^2)$ , and each discrete  $x^2$  corresponds to a different sigma model. The system (1.1) is a collection of parallel  $(1+1)$ -dimensional  $SU(N) \times SU(N)$  sigma models, each of which couples to the auxiliary fields  $A_0, A_1$ . The sigma-model self-interaction is the dimensionless number  $g_0$ .

We feel it worth commenting on the nature of the anisotropic regime and how it is different from the standard  $(2+1)$ -dimensional Yang-Mills theory. The point where the regulator can be removed in the theory is  $g'_0 = g_0 = 0$ . This point can be reached in our treatment, but only if

$$(g'_0)^2 \ll \frac{1}{g_0} e^{-4\pi/(g_0^2 N)} . \quad (1.2)$$

The left-hand side and right-hand side are proportional to the two energy scales in the theory (the latter comes from the two-loop beta function of the sigma model). Thus our method cannot accommodate fixing the ratio  $g'_0/g_0$ , which is natural in standard perturbation theory [11]. This is why the mass gap is not of order  $e$ ,  $e'$  and the string tension is not of order  $e^2$ ,  $(e')^2$ .

We now discuss the Hamiltonian in the axial gauge  $A_1 = 0$ . The left-handed and right-handed currents are,  $j_\mu^L(x)_b = i\text{Tr} t_b \partial_\mu U(x) U(x)^\dagger$  and  $j_\mu^R(x)_b = i\text{Tr} t_b U(x)^\dagger \partial_\mu U(x)$ , respectively, where  $\mu = 0, 1$ . The Hamiltonian obtained from (1.1) is  $H_0 + H_1$ , where

$$H_0 = \sum_{x^2} \int dx^1 \frac{1}{2g_0^2} \{ [j_0^L(x)_b]^2 + [j_1^L(x)_b]^2 \} , \quad (1.3)$$

and

$$\begin{aligned}
H_1 = & \sum_{x^2} \int dx^1 \frac{(g'_0)^2 a^2}{4} \partial_1 \Phi(x^1, x^2) \partial_1 \Phi(x^1, x^2) \\
& - \left( \frac{g'_0}{g_0} \right)^2 \sum_{x^2=0}^{L^2-a} \int dx^1 [j_0^L(x^1, x^2) \Phi(x^1, x^2) - j_0^R(x^1, x^2) \Phi(x^1, x^2 + a)] \\
& + (g'_0)^2 q_b \Phi(u^1, u^2)_b - (g'_0)^2 q'_b \Phi(v^1, v^2)_b,
\end{aligned} \tag{1.4}$$

where  $-\Phi_b = A_0{}_b$  is the temporal gauge field, and where in the last term we have inserted two color charges - a quark with charge  $q$  at site  $u$  and an anti-quark with charge  $q'$  at site  $v$ . Some gauge invariance remains after the axial-gauge fixing, namely that for each  $x^2$

$$\left\{ \int dx^1 [j_0^L(x^1, x^2)_b - j_0^R(x^1, x^2 - a)_b] - g_0^2 Q(x^2)_b \right\} \Psi = 0, \tag{1.5}$$

where  $Q(x^2)_b$  is the total color charge from quarks at  $x^2$  and  $\Psi$  is any physical state. To derive the constraint (1.5) more precisely, we started with open boundary conditions in the 1-direction and periodic boundary conditions in the 2-direction, meaning that the two-dimensional space is a cylinder [1], [2].

From (1.4) we see that the left-handed charge of the sigma model at  $x^2$  is coupled to the electrostatic potential  $\Phi$ , at  $x^2$ . The right-handed charge of the sigma model is coupled to the electrostatic potential at  $x^2 + a$ . The excitations of  $H_0$ , which we call Fadeev-Zamolodchikov or FZ particles, behave like solitons, though they do not correspond to classical configurations. Some of these FZ particles are elementary and others are bound states of the elementary FZ particles. An elementary FZ particle has an adjoint charge and mass  $m_1$ . An elementary one-FZ-particle state is a superposition of color-dipole states, with a quark (anti-quark) charge at  $x^1, x^2$  and an anti-quark (quark) charge at  $x^1, x^2 + a$ . The interaction  $H_1$  produces a linear potential between color charges with the same value of  $x^2$ . Residual gauge invariance (1.5) requires that at each value of  $x^2$ , the total color charge is zero. If there are no quarks with coordinate  $x^2$ , the total right-handed charge of FZ particles in the sigma model at  $x^2 - a$  is equal to the total left-handed charge of FZ particles in the sigma model at  $x^2$ .

The particles of the principal chiral sigma model carry a quantum number  $r$ , with the values  $r = 1, \dots, N - 1$  [21]. Each particle of label  $r$  has an antiparticle of the same mass, with label  $N - r$ . The masses are given by

$$m_r = m_1 \frac{\sin \frac{r\pi}{N}}{\sin \frac{\pi}{N}}, \quad m_1 = K \Lambda (g_0^2 N)^{-1/2} e^{-\frac{4\pi}{g_0^2 N}} + \text{non-universal corrections}, \tag{1.6}$$

where  $K$  is a non-universal constant and  $\Lambda$  is the ultraviolet cut-off of the sigma model.

Lorentz invariance in each  $x^0, x^1$  plane is manifest. For this reason, the linear potential is not the only effect of  $H_1$ . The interaction creates and destroys pairs of

elementary FZ particles. This effect is quite small, provided that  $g'_0$  is small enough. Specifically, this means that the square of the  $1 + 1$  string tension in the  $x^1$ -direction coming from  $H_1$  is small compared to the square of the mass of fundamental FZ particle; this is just the condition (1.2). The effect is important, however, in that it is responsible for the correction to the horizontal string discussed in the next paragraph in equation (1.8).

Simple arguments readily show that at leading order in  $g'_0$ , the vertical and horizontal string tensions are given by

$$\sigma_V = \frac{m_1}{a} , \quad \sigma_H = \frac{(g'_0)^2}{2a^2} C_N , \quad (1.7)$$

respectively, where  $C_N$  is the smallest eigenvalue of the Casimir of  $SU(N)$ . These naive results for the string tension have further corrections in  $g'_0$ , which were determined for the horizontal string tension for  $SU(2)$  [2]:

$$\sigma_H = \frac{3}{2} \left( \frac{g'_0}{a} \right)^2 \left[ 1 + \frac{4}{3} \frac{0.7296}{K^2 \pi^2} \Lambda a \frac{(g'_0)^2}{g_0^2} e^{4\pi/g_0^2} \right]^{-1} . \quad (1.8)$$

The leading term agrees with (1.7). This calculation was done using the exact form factor for sigma model currents obtained by Karowski and Weisz [12]. The form factor can also be used to find corrections of order  $(g'_0)^2$  to the vertical string tension; this problem should be solved soon. If the reader is not familiar with form-factor techniques in relativistic integrable field theories, a self-contained review is in the appendix of reference [2].

Another recent application of exact form factors to the  $(2 + 1)$ -dimensional  $SU(2)$  gauge theory is reference [13], in which form factors of the two-dimensional Ising model [14] are used to find the profile of the electric string near the high-temperature deconfining transition, assuming the Svetitsky-Yaffe conjecture [15].

A rough picture of a gauge-invariant state for the gauge group  $SU(2)$  with no quarks is given in Figure 1. For  $N > 2$ , there are more complicated ways in which strings can join particles. For example, a junction of  $N$  strings is possible. Figure 1 is inaccurate in an important respect; the “ring” of particles held together by horizontal strings is extremely broad in extent in the  $x^2$ -direction compared to the  $x^1$ -direction. This is because  $\sigma_H \ll \sigma_V$ .

The lightest states have the smallest number of particles, by virtue of  $\sigma_H \ll \sigma_V$ . Thus the lightest glueballs are pairs of FZ particles with the same value of  $x^2$ . For small enough  $g'_0$ , the very lightest state has a mass well-approximated by  $2m_1$ . The purpose of this paper is to find the leading corrections in  $(g'_0)^2$  to this result. This will be done using the S-matrix of the sigma model and the WKB formula. There are further small corrections, due to the softening of the potential near where particles overlap, which we do not determine.

It is clear that the lightest bound states of FZ particles are  $(1 + 1)$ -dimensional in character. If we formulated a gauge theory in which  $x^2$  was fixed in  $U(x^0, x^1, x^2)$ , we

would find the same spectrum, as a function of  $m_1$  and  $\sigma_H$ . In the Kogut-Susskind lattice formulation, a long row of plaquettes with open boundary conditions is a regularized gauge theory of this type. The only real difference between this  $(1+1)$  dimensional model and that we study is that  $\sigma_H$  will receive different corrections of order  $(g'_0)^2$ .

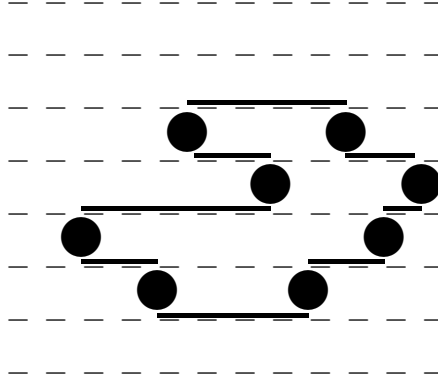


Figure 1. A glueball state is a collection of heavy particles, held weakly together by strings. The horizontal coordinate is  $x^1$  and the vertical coordinate is  $x^2$ .

In the next section we will discuss the wave function of an unbound pair of FZ particles. We find that this is described by phase shift for the color-singlet sector. In Section 3, we determine the bound-state spectrum. The problem we solve is very similar to that of two particle-states of the two-dimensional Ising model with an external magnetic field [16] (for a good summary of this problem, see reference [17]); the only genuine difference is the presence of a matching condition where the particles overlap. This matching condition comes from the phase shift of the scattering problem. We present our conclusions in Section 4.

## 2 Scattering states of FZ particles

The lightest glueball state, as discussed above, is simply a pair of FZ particles located at the points  $(x^1, x^2)$  and  $(y^1, x^2)$  and bound in a linear potential. Residual gauge invariance (1.5), demands that the state be a color singlet. To begin with, however, we simply write the form of a free state of two particles.

The state of the  $SU(2) \times SU(2) \simeq O(4)$  nonlinear sigma model with a particles of momenta  $p_1$  and  $p_2$  and quantum numbers  $j_1$  and  $j_2$  (which take the values  $1, 2, 3, 4$ ) is described by the wave function

$$\psi_{p_1 p_2}(x^1, y^1)_{j_1, j_2} = \begin{cases} e^{ip_1 x^1 + ip_2 y^1} A_{j_1, j_2} , & x^1 < y^1 \\ e^{ip_2 x^1 + ip_1 y^1} \sum_{k_1, k_2=1}^4 S_{j_1 j_2}^{k_1 k_2}(p_1, p_2) A_{k_2, k_1} , & x^1 > y^1 \end{cases} , \quad (2.1)$$

where  $A_{j_1 j_2}$  is an arbitrary set of complex numbers and  $S_{j_1 j_2}^{k_1 k_2}(p_1, p_2)$  is the two-particle S-matrix. We have not yet imposed (1.5).

The wave function (2.1) is written in a form where the  $O(4)$  symmetry is manifest. It is straightforward to write it in a form where the left  $SU(2)_L$  and the right  $SU(2)_R$  symmetries are manifest, by writing

$$\psi_{p_1 p_2}(x^1, y^1)_{a, \bar{b}}^{\bar{c}, d} = \sum_{j_1, j_2} \frac{1}{\sqrt{2}} (\delta_{ac}^{j_1 4} - i\sigma_{ac}^{j_1}) \frac{1}{\sqrt{2}} (\delta_{bd}^{j_2 4} - i\sigma_{bd}^{j_2})^* \psi_{p_1 p_2}(x^1, y^1)_{j_1, j_2} \quad (2.2)$$

describing a pair of color dipoles, one with quantum numbers  $a, \bar{b}$  and the other with quantum numbers  $\bar{c}, d$ , where  $\sigma^j$ ,  $j = 1, 2, 3$  denotes the Pauli matrices.

We impose the physical state condition (1.5) on (2.2) by requiring that  $a = b$  and  $c = d$  and summing over these colors. The projected wave function is, up to an overall constant,

$$\psi_{p_1 p_2}(x^1, y^1) = \begin{cases} e^{ip_1 x^1 + ip_2 y^1}, & x^1 < y^1 \\ e^{ip_2 x^1 + ip_1 y^1} S_0(p_1, p_2), & x^1 > y^1 \end{cases}, \quad (2.3)$$

where  $S_0(p_1, p_2)$  is the singlet projection of the  $O(4)$  S-matrix. This S-matrix was first obtained by Zamolodchikov and Zamolodchikov [18]. A useful form is given in reference [12]:

$$S_0(p_1, p_2) = S_0(\theta) = -\frac{\pi - i\theta}{\pi + i\theta} \exp i \int_0^\infty \frac{d\xi}{\xi} \frac{1 - e^{-\xi}}{1 + e^\xi} \sin \frac{\xi\theta}{\pi}, \quad (2.4)$$

where the relative rapidity  $\theta$  is given by  $\theta = \theta_2 - \theta_1$ ,  $p_1 = m \sinh \theta_1$ ,  $p_2 = m \sinh \theta_2$  and where we denote the particle mass  $m_1$ , given by (1.6), by  $m$  (because there is only one mass for the case of  $N = 2$ ). A derivation of (2.4) is in the appendix of reference [2].

The singlet S-matrix is just given by a phase shift  $\phi(\theta)$ :  $S_0(\theta) = \exp i\phi(\theta)$ . The phase shift has a simple form in the low-energy, non-relativistic limit,  $|p_1 - p_2| \ll m$ . In this limit,  $\theta \approx |p_1 - p_2|/m$ . The integral on the right-hand side of (2.4) can be done by Taylor expanding in  $|p_1 - p_2|/m$  yielding

$$\phi(\theta) = \phi(p_1, p_2) = \pi - \frac{3 - 2 \ln 2}{\pi m} |p_1 - p_2| + O\left(\frac{|p - r|^2}{m^2}\right). \quad (2.5)$$

### 3 The low-lying glueball spectrum

Let us now consider the states of a bound pair of FZ particles in the potential  $V(x^1, y^1) = 2\sigma_H |x^1 - y^1|$  (the reason for the factor of two is simply that the particles are joined by a pair of strings). We use the non-relativistic approximation, used to find (2.5). For our problem, the horizontal string tension times the size of a typical bound state is

small compared to the mass, by (1.2). This justifies the non-relativistic approximation for low-lying states. The mass of a low-lying glueball is given by

$$M = 2m + E ,$$

where  $E$  is the energy eigenstate of the two-particle problem.

Let us introduce center-of-mass coordinates,  $X = (x^1 + y^1)/2$  and  $x = y^1 - x^1$ . The reduced mass of the system is  $m/2$ . We factor out the phase depending on  $X$ , leaving us only with a wave function depending on  $x$ . The Schrödinger equation we consider is

$$-\frac{1}{m} \frac{d^2 \psi}{dx^2} + 2\sigma_H |x| \psi = E \psi$$

with a matching condition at  $x = 0$  between the wave function  $\psi(x)$  at  $x > 0$  and the wave function at  $x < 0$ . There is actually a further complication, which we do not consider here; the potential changes slightly in the region where  $x \approx 0$ . This is due to the fact that the color charge is slightly smeared out. This smearing out can be calculated from the form factor [12].

Our results (2.3), (2.5) for the unbound two-particle state, tell us that for  $x^1 \approx y^1$ , where the effect of the potential can be ignored, the bound-state wave function in the center-of-mass frame will be of the form

$$\psi(x) = \begin{cases} \cos(px + \omega) , & x < 0 \\ \cos[-px + \omega - \phi(p)] , & x > 0 \end{cases} , \quad (3.1)$$

for some angle  $\omega$ , where  $p = p_1 - p_2$  and  $\phi(p) = \pi - \frac{3-2\ln 2}{\pi m} |p| + O(|p|^2/m^2)$ . The value of  $p$  near  $x = 0$  is given by  $p = (mE)^{1/2}$ , where  $E$  is the energy eigenvalue of the state. This is the matching condition between the wave function for  $x > 0$  and for  $x < 0$ .

The wave function for  $x < 0$  is an Airy function. So is the wave function for  $x > 0$ . We therefore obtain the approximate WKB form

$$\psi(x) = \begin{cases} C(x + \frac{E}{2\sigma_H})^{-1/4} \cos \left[ \frac{2}{3} (2m\sigma_H)^{1/2} (x + \frac{E}{2\sigma_H})^{3/2} - \frac{\pi}{4} \right] , & x < 0 \\ C'(\frac{E}{2\sigma_H} - x)^{-1/4} \cos \left[ -\frac{2}{3} (2m\sigma_H)^{1/2} (\frac{E}{2\sigma_H} - x)^{3/2} + \frac{\pi}{4} \right] , & x > 0 \end{cases} , \quad (3.2)$$

for some constants  $C$  and  $C'$ . The expression (3.2) can be made to agree with (3.1) for small  $x$ , provided the generalization of the Bohr-Sommerfeld quantization condition

$$\frac{2(m)^{1/2}}{3\sigma_H} E_n^{3/2} + \frac{3-2\ln 2}{\pi m^{1/2}} E_n^{1/2} - \left( n + \frac{1}{2} \right) \pi = 0 , \quad n = 0, 1, 2, \dots , \quad (3.3)$$

is satisfied by  $E = E_n$ . The only new feature in this semi-classical formula is the second term, produced by the phase shift. Absorbing the horizontal string tension in the energy, by defining  $u_n = E_n \sigma_H^{-2/3}$ , this cubic equation becomes

$$\frac{2(m)^{1/2}}{3} u_n^{3/2} + \frac{3-2\ln 2}{\pi m^{1/2}} \sigma_H^{1/3} u_n^{1/2} - \left( n + \frac{1}{2} \right) \pi = 0 .$$



The second term can be ignored for sufficiently small  $\sigma_H$ , i.e. sufficiently small  $g'_0$ .

There is a unique real solution of the cubic equation (3.3) for a given integer  $n \geq 0$ , because  $3 - 2 \ln 2 = 1.613706 > 0$ . The low-lying glueball masses are given by

$$M_n = 2m + E_n = 2m + \left[ \epsilon_n^{1/3} - \frac{3(3 - 2 \ln 2)\sigma_H}{4\pi m} \epsilon_n^{-1/3} \right]^2, \quad (3.4)$$

where

$$\epsilon_n = \frac{3\pi\sigma_H(n + \frac{1}{2})}{4m^{1/2}} + \left\{ \left[ \frac{3\pi\sigma_H}{4m^{1/2}(n + \frac{1}{2})} \right]^2 + \frac{1}{8} \left[ \frac{3(3 - 2 \ln 2)\sigma_H}{2\pi m} \right]^3 \right\}^{1/2}. \quad (3.5)$$

## 4 Conclusions

We have identified the low-lying glueballs of the anisotropic Yang-Mills theory in  $(2 + 1)$  dimensions as bound pairs of the fundamental massive particles of the principal chiral nonlinear sigma model. We found a matching condition for the bound-state wave function at the origin, which when combined with elementary methods yields the spectrum of the lightest states.

There are other aspects of the two-particle bound-state problem we have not considered here. First, the potential is not precisely linear in the region where the two particles are close together. The corrections to the potential can be determined using form factors. This will slightly modify (3.3). A completely different issue is that there are small corrections to the form factors themselves, coming from the presence of bound states. This, in turn, will give a further correction to the horizontal string tension found in [2]. Such corrections to form factors in theories close to integrability were first discussed by Delfino, Mussardo and Simonetti [19]. The bound-state energies proliferate between  $2m$  and  $4m$ , as  $g'_0 \rightarrow 0$ . Our method breaks down as the bound-state mass reaches  $4m$ , because the bound state develops an instability towards fission into a pair of two-particle bound states. This is analogous to the situation for the Ising model in a field [16], [17] as we stated earlier. It should be worthwhile to understand the relativistic corrections to the bound-state formula, along the lines of the work of Fonseca and Zamolodchikov [20].

A similar calculation is possible for  $SU(N)$ . The exact S-matrix of the principal chiral nonlinear sigma model is known for  $N > 2$  [21]. An interesting feature is that the phase shift should vanish as  $N \rightarrow \infty$ , with  $g_0^2 N$  fixed, meaning that the wave function would be continuous where FZ particles overlap.

It would be interesting to study the scattering of a glueball by an external particle. If the scattering is sufficiently short range, the FZ particles could be liberated from the glueball, after which hadronization would ensue.

The results of this paper and of references [1] and [2] may be extendable to the standard  $(2 + 1)$ -dimensional isotropic Yang-Mills theory with  $g'_0 = g_0$ . The strategy we have in mind is an anisotropic renormalization procedure. At the start is a

standard field theory with an isotropic cut-off. By anisotropically integrating out high-momentum degrees of freedom, the isotropic theory will flow to an anisotropic theory with a small momentum cut-off in the  $x^2$ -direction and a large momentum cut-off in the  $x^1$  direction. If the renormalized couplings satisfy the condition (1.2), we could apply our techniques. A check of such a method would be approximate rotational invariance of the string tension. This would give an analytic first-principles method of solving the isotropic gauge theory with fixed dimensionful coupling constant  $e$ , and no cut-off. The only other analytic weak-coupling argument for a mass gap and confinement in  $(2+1)$ -dimensions, namely that of orbit-space distance estimates, discussed by Feynman [22], by Karabali and Nair in the second of references [5], and by Semenoff and the author [23] is suggestive, but has not yielded definite results yet<sup>1</sup>.

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<sup>1</sup>See also reference [24] for a general discussion of distance in orbit space.

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